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CALCULATION OF NONLINEAR TRANSIENT MOTION OF CABLES

Thomas S. Walton, et al.

David Taylor Model Basin
Washington, D. C.

July 1959

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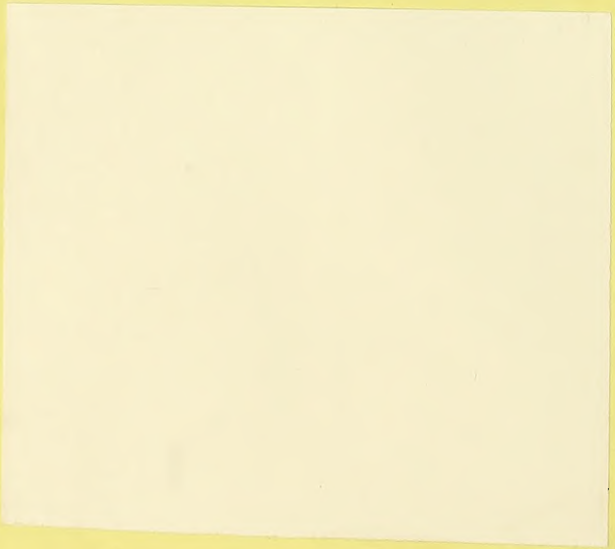

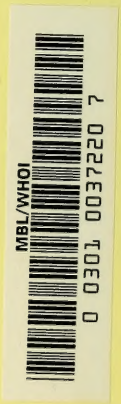

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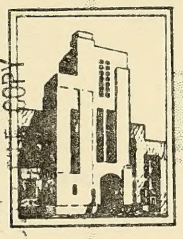

DEPARTMENT OF THE NAVY

DAVID TAYLOR MODEL BASIN

HYDROMECHANICS

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APPLIED MATHEMATICS

CALCULATION OF NONLINEAR TRANSIENT MOTION OF CABLES

by

Thomas S. Walton and Harry Polachek

APPLIED MATHEMATICS LABORATORY

RESEARCH AND DEVELOPMENT REPORT

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NOTATION

a	Acceleration
$C^D_{j+1/2}$	Drag coefficient for segment of cable between stations j and j + 1
C^X_j	Resistance coefficient for horizontal motion of suspended prism
C^Y_j	Resistance coefficient for vertical motion of suspended prism
c	Velocity of uniform horizontal current
D	Drag
$d_{j+1/2}$	Diameter of segment of cable between stations j and j + 1
$e_{j+1/2}$	Virtual mass of entrained fluid between stations j and j + 1
F	Resultant force
$f^D_{j+1/2}$	Drag factor for cable $= (\rho/2)\, C^D_{j+1/2}\, \ell_{j+1/2}\, d_{j+1/2}$
f^X_j	Horizontal drag factor for suspended prism $= (\rho/2)\, C^X_j\, S^X_j$
f^Y_j	Vertical drag factor for suspended prism $= (\rho/2)\, C^Y_j\, S^Y_j$
g	Acceleration due to gravity
I_j	Component of inertia tensor
i	Imaginary unit
J_j	Component of inertia tensor
j	Subscript denoting station number along line

K_j	Component of inertia tensor
$k_{j+1/2}$	Virtual inertia coefficient for segment of cable between stations j and j + 1
$l_{j+1/2}$	Length of line between stations j and j + 1
m_j	Mean mass of segments of cable adjoining station j
m_j^*	Mass of prism suspended from station j
m_j^X	Effective horizontal mass of suspended prism
m_j^Y	Effective vertical mass of suspended prism
n	Superscript denoting time-step number
o	Superscript denoting initial state (origin in time), or subscript denoting anchor end of line
p	Tangential component of velocity of cable (relative to medium)
q	Normal component of velocity of cable (relative to medium)
S_j^X	Projected area of suspended prism along x-axis
S_j^Y	Projected area of suspended prism along y-axis
s	Subscript denoting surface end of line
T	Tension
t	Time
Δt	Time-step interval
u	Magnitude of velocity of cable (relative to medium)
V_j^*	Volume of prism suspended from station j
V_j^X	Equivalent volume of horizontal virtual mass of suspended prism

V_j^Y	Equivalent volume of vertical virtual mass of suspended prism
W_j	Mean net weight of segments of cable adjoining station j
W_j^*	Net weight of prism suspended from station j
X	Horizontal component of resultant external force
X_j^*	Horizontal component of damping force on suspended prism
x	Horizontal coordinate of cable
Y	Vertical component of resultant external force
Y_j^*	Vertical component of damping force on suspended prism
y	Vertical coordinate of cable
α	Damping coefficient of the perturbation functions
β	Dimensionless frequency of the perturbation functions
δ	The variation of
θ	Angle between horizontal and tangent to cable
λ	Eigenvalue (root of characteristic equation)
$\mu_{j+1/2}$	Linear density of segment of cable between stations j and j + 1
ρ	Density of fluid medium
$\sigma_{j+1/2}$	Cross-section area of segment of cable between stations j and j + 1
$\dot{}$	Dot signifies differentiation with respect to time
$\tilde{}$	Tilde signifies tentative value of a variable

ABSTRACT

The system of partial differential equations governing the nonlinear transient motion of a cable immersed in a fluid is solved by finite difference methods. This problem may be considered a generalization of the classical vibrating string problem in the following respects: a) the motion is two dimensional, b) large displacements are permitted, c) forces due to the weight of the cable, buoyancy, virtual inertia of the medium and damping or drag are included, and d) the cable is assumed to be nonuniform. The numerical solution of this system of equations presented a number of interesting mathematical problems related to: a) the nonlinear nature of the equations, b) the determination of a stable numerical procedure, and c) the determination of an effective computational method. The computation is programmed for a high-speed calculator (UNIVAC system). The solution of this problem is of practical significance in the calculation of the transient forces acting on mooring lines due to the bobbing up and down of ships during the period preceding large scale explosion tests, as well as in many other applications involving mooring or towing operations.

1. INTRODUCTION

This problem arose as a result of an urgent requirement by the Navy in connection with a series of nuclear explosion tests which were conducted in the Pacific. In preparation for these tests a number of ships were instrumented and moored at specified locations from the explosion point. These positions had to be maintained intact during the period preceding the explosion. However, the bobbing up and down of the ships due to ocean waves during this period could excite sizeable transient forces in the mooring lines which might break these lines and thus result in the loss of information from the tests. Several months prior to these tests a request was made to the Applied Mathematics Laboratory to calculate the magnitude of the forces acting on the mooring lines, for waves of varying amplitude and frequency. It is gratifying to report that in spite of the theoretical complexities of this problem and the absence of any known solutions, the Applied Mathematics Laboratory was able to obtain the required results in time for use during the scheduled tests. The two factors which made a theoretical solution feasible at this time, whereas it would not have been possible several years ago, were: a) the availability of a high-speed computer and b) the recent progress made in the understanding and development of numerical methods for the solution of systems of partial differential equations.

Whereas the solution to this problem was carried out as a result of one specific requirement, it is more useful to regard it as the general problem of the two-dimensional motion of a cable or rope immersed in a

fluid. From this point of view it may be considered as a rather broad generalization of the classical vibrating string problem, and it becomes immediately apparent that its solution is applicable to a wide class of engineering problems involving the motion of cables, such as a) the laying of intercontinental telegraph cables, b) the towing of a ship or other object in water, or c) the snapping of telephone wires as a result of transient forces caused by storm. This problem may be stated abstractly as follows: Given the initial conditions (i.e., position and velocity at any time, t_0) and boundary conditions (positions of end points at all times) of a cable immersed in a fluid, determine its subsequent motions. The motions are assumed to take place in two dimensions.

Forces that are assumed acting on the cable are: a) forced motion of the extremities (end points) of the cable, b) damping or drag as it moves through the fluid, c) virtual inertia due to the motion imparted to the fluid d) weight of the cable, and e) buoyancy. Variations in the mass as well as other physical properties of the cable along its length are allowed. The displacements may be large and the motions rapid. In the present solution it is assumed that the cable is inextensible (cannot be stretched). In subsequent work the authors have carried out solutions for cables with elastic properties. The motions are not restricted in any manner (except that these take place in two dimensions).

The solution was carried out by the method of finite differences. This method consists simply in replacing the derivatives of various order

in the differential system of equations by equivalent ratios of finite increments. This substitution results in a system of difference equations, which are algebraic in form, and hence more easily tractable. However, in order to represent a valid solution, the system of finite difference equations so derived must possess certain mathematical stability (and convergence) properties. It must have the property that its solution progressively increases in accuracy as the size of the time increment used in the above representation is gradually decreased. Unfortunately, the system of finite difference equations initially proposed for the solution of this problem did not satisfy these stability requirements. A search for a stable finite difference system as well as for an effective method for solving the resulting system of finite difference equations, which was nonlinear in character, added to the complexity of the problem.

2. DERIVATION OF EQUATIONS OF MOTION

The problem under consideration is a generalization of the classical problem of the motion of a vibrating string. In at least one respect the formulation of the problem given here will not differ from that of the original. Specifically, we wish to deduce the approximate motion of a steel cable without having to involve ourselves in the explicit computation of the elastic forces which act on the cable. However, the formulation will depart from the original in a number of respects, namely:

a) Longitudinal as well as transverse motions of the line must be taken into consideration.

b) The occurrence of large displacements from the equilibrium configuration of the line must be permitted.

c) The weight of the cable must be taken into account because the line may stretch from one level to another. Thus, even when the line is in static equilibrium, the tension will not be uniform nor will the line be straight.

d) Since the cable is submerged, the static forces must include the buoyancy of the medium and the dynamic forces must allow for the virtual inertia of the medium. Furthermore, it is desired to make provision for damping forces due to the drag on the line whenever lateral motion is occurring.

e) Finally, it is desired to suspend concentrated loads at one or more points along the line and to change the linear density of the cable at specified points.

The best approach to the solution of a problem with such general specifications appears to be a numerical method based on finite difference approximations. Inasmuch as we are committing ourselves to the eventual use of differences in both the time and space dimensions, it will be simpler to introduce the spacewise discreteness into the original formulation of the problem. We therefore proceed at once to the derivation of the equation of motion of a simplified model in which the distributed mass of the cable has been replaced by a series of discrete masses m_j attached to a weightless, inextensible line. This leads to a system of ordinary differential equations. It may be shown that, in the limit, the resulting equations pass over into the corresponding partial differential equations for the motion of a submerged cable.

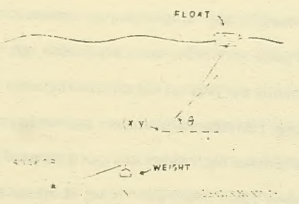

Figure 1 - Mooring line

Figure 2 - Discrete representation

Figure 1 shows a typical configuration of the system with the cable attached to a float at the surface and anchored to the bottom. Also, a heavy load is suspended from a point near one end of the cable. Other boundary conditions are possible, but the equations of motion will be the same in any case. The horizontal and vertical coordinates of a point on the line are called x and y, respectively, and the angle between the horizontal and the tangent to the line is designated by θ. Figure 2 illustrates the corresponding discrete model for which the equations will actually be derived. The line is divided into segments in such a way that there will always be an integral number of them between any points where an abrupt change occurs in some parameter. The junctions between the segments are numbered according to the subscript index j, which runs from 0 at the anchor to s at the surface.

Before we can properly invoke Newton's law of motion, it is necessary to consider the inertial properties of the fluid in which the cable is immersed. We shall assume that the kinetic energy of the surrounding medium is independent of the component of velocity parallel to the line, whereas it varies as the square of the component of velocity at right angles to the line. Thus, when an element of the cable is accelerated longitudinally, no hydrodynamic reaction occurs, but when the cable is accelerated laterally, it behaves as though it possessed additional inertia. The component of acceleration normal to the line is

$$a^{\text{normal}} = -\ddot{x} \sin\theta + \ddot{y} \cos\theta$$

The accompanying inertial reaction can be resolved into horizontal and vertical components. Each of these will be proportional to the corresponding component of a$^{\text{normal}}$, namely,

HORZ. COMP: $-a^{\text{normal}} \sin\theta = \ddot{x} \sin^2\theta - \ddot{y} \sin\theta \cos\theta$

VERT. COMP: $a^{\text{normal}} \cos\theta = \ddot{y} \cos^2\theta - \ddot{x} \sin\theta \cos\theta$

Thus, each component of the hydrodynamic reaction depends on both components of acceleration. In general, the reaction force is not parallel to the acceleration vector (except when the tangential component is zero), so that it is necessary to regard the inertial parameters of the system as tensors rather than simple scalars.

The differential equations governing the motion of the j$^{\text{th}}$ station on the line (see Fig. 2) can be written in matrix notation as follows:

$$\begin{Bmatrix} I_j - K_j \\ -K_j \quad J_j \end{Bmatrix} \begin{Bmatrix} \ddot{x}_j \\ \ddot{y}_j \end{Bmatrix} = \begin{Bmatrix} F_j^{\text{HORZ}} \\ F_j^{\text{VERT}} \end{Bmatrix} \tag{2.1}$$

where: $I_j = m_j + \frac{1}{2}\left(e_{j+\frac{1}{2}} \sin^2\theta_{j+\frac{1}{2}} + e_{j-\frac{1}{2}} \sin^2\theta_{j-\frac{1}{2}}\right) + m_j^X$

$J_j = m_j + \frac{1}{2}\left(e_{j+\frac{1}{2}} \cos^2\theta_{j+\frac{1}{2}} + e_{j-\frac{1}{2}} \cos^2\theta_{j-\frac{1}{2}}\right) + m_j^Y$

$K_j = \frac{1}{2}\left(e_{j+\frac{1}{2}} \sin\theta_{j+\frac{1}{2}} \cos\theta_{j+\frac{1}{2}} + e_{j-\frac{1}{2}} \sin\theta_{j-\frac{1}{2}} \cos\theta_{j-\frac{1}{2}}\right)$

and $m_j = \frac{1}{2}\left[\mu_{j+\frac{1}{2}}\, l_{j+\frac{1}{2}} + \mu_{j-\frac{1}{2}}\, l_{j-\frac{1}{2}}\right]$

$e_{j+\frac{1}{2}} = \rho k_{j+\frac{1}{2}}\, l_{j+\frac{1}{2}}\, \sigma_{j+\frac{1}{2}}$

$m_j^X = m_j^* + \rho V_j^X$

$m_j^Y = m_j^* + \rho V_j^Y$

$\cos\theta_{j+\frac{1}{2}} = (x_{j+1} - x_j)/l_{j+\frac{1}{2}}$

$\sin\theta_{j+\frac{1}{2}} = (y_{j+1} - y_j)/l_{j+\frac{1}{2}}$

Each lumped mass, m_j, has been expressed as the average mass of the two segments of cable which lie on either side of station j. Also, one-half the equivalent transverse mass, e_j, of the entrained fluid associated with each of these segments has been included in the inertia tensor. Furthermore, at those stations from which a weight is suspended, the effective horizontal and vertical masses, m_j^X and m_j^Y, of the weight are to be added. For simplicity in allowing for virtual inertia, we have assumed that any such weights possess a certain degree of symmetry and remain upright as the line moves about.

The force vector, F_j, on the right side of eq. (2.1) can be expressed as the sum of internal forces (the tensions acting between adjacent mass elements) plus whatever external forces are present. Thus, in expanded form the equations of motion can be written

$$I_j \ddot{x}_j - K_j \ddot{y}_j - T_{j+1/2} \cos\theta_{j+1/2} - T_{j-1/2} \cos\theta_{j-1/2} + X_j$$

$$- K_j \ddot{x}_j + J_j \ddot{y}_j = T_{j+1/2} \sin\theta_{j+1/2} - T_{j-1/2} \sin\theta_{j-1/2} + Y_j \qquad (2.2)$$

where $T_{j+1/2}$ = tension in segment of line between stations j and j + 1

X_j = horizontal component of resultant external force at station j

Y_j = vertical component of resultant external force at station j

There are two sources of external force, namely: 1) gravity, which gives rise to the weight minus the buoyancy and acts only in the vertical, and 2) fluid resistance, which gives rise to the damping forces. Thus, we write

$$X_j = -\frac{1}{2}\left[D_{j+\frac{1}{2}} \sin\theta_{j+\frac{1}{2}} + D_{j-\frac{1}{2}} \sin\theta_{j-\frac{1}{2}}\right] + X_j^*$$

$$Y_j = \frac{1}{2}\left[D_{j+\frac{1}{2}} \cos\theta_{j+\frac{1}{2}} + D_{j-\frac{1}{2}} \cos\theta_{j-\frac{1}{2}}\right] + Y_j^* - W_j - W_j^* \qquad (2.3)$$

where $W_j = m_j g - \frac{1}{2}\rho g \left(l_{j+\frac{1}{2}} \sigma_{j+\frac{1}{2}} + l_{j-\frac{1}{2}} \sigma_{j-\frac{1}{2}}\right)$

$W_j^* = m_j^* g - \rho g V_j^*$

and $D_{j+\frac{1}{2}}$ = drag on segment of line between stations j and j + 1

X_j^* = horizontal component of damping force on weight at station j

Y_j^* = vertical component of damping force on weight at station j

Again, in order to get the best approximation to the continuous case, the net effect of the drag at station j has been expressed as one-half of each component of the drag on the segments which lie on either side of this station. The buoyant force on the displaced fluid has been treated likewise.

We have assumed that the drag, $D_{j+\frac{1}{2}}$, on a segment of the line acts in a direction at right angles to the line. This is a good approximation whenever the velocity is high enough to produce significant forces, since at all but the lowest Reynolds numbers the tangential component of the hydrodynamic force is very small compared to the normal component. Furthermore, we assume that the drag is proportional to the square of the component of relative velocity normal to the line:

$$D_{j+1/2} = -f^D_{j+1/2}\, q_{j+1/2}\, |q_{j+1/2}| \qquad (2.4)$$

where $f^D_{j+1/2} = \frac{1}{2}\rho C^D_{j+1/2}\, l_{j+1/2}\, d_{j+1/2}$

$$q_{j+1/2} = -\frac{1}{2}\left[(\dot{x}_{j+1} - c) + (\dot{x}_j - c)\right] \sin\theta_{j+1/2} + \frac{1}{2}\left[\dot{y}_{j+1} + \dot{y}_j\right] \cos\theta_{j+1/2}$$

The positive normal to the line has been arbitrarily taken to be directed upward when θ equals zero. The introduction of the minus sign and the use of the absolute value of one of the velocity factors ensures that the drag will always be opposed to the direction of $q_{j+1/2}$ and thus act as a dissipative force to remove energy from the system. Since the velocities of the two endpoints of each segment will, in general, differ slightly, their mean value (which for a straight line segment is exactly equal to the velocity of the midpoint) is taken as a representative value in the definition of $q_{j+1/2}$. In addition, the definition allows for the presence of a uniform horizontal current, c, to incorporate the ability to treat towing lines as well as mooring lines (or mooring lines subjected to ocean currents).

In addition to the drag on the line itself, there will also be resistance to the motion of any concentrated loads which may be suspended from the line. These additional damping forces will vary with the velocity but will not, in general, be directed exactly opposite to the motion of each weight. However, on account of the assumed orientation and symmetry of any such weights, the resistance force will be parallel to the velocity vector whenever the relative motion is either purely horizontal or purely vertical. Accordingly, the two components of resistance may be written

$$\begin{aligned} X_j^* &= - f_j^X u_j (\dot{x}_j - c) \\ Y_j^* &= - f_j^Y u_j \dot{y}_j \end{aligned} \qquad (2.5)$$

where: $f_j^X = \frac{1}{2} \rho C_j^X S_j^X$

$f_j^Y = \frac{1}{2} \rho C_j^Y S_j^Y$

$u_j = \left[(\dot{x}_j - c)^2 + \dot{y}_j^2 \right]^{1/2}$

Up to this point an explicit formula has been given for the evaluation of every term in the equations of motion (2.2) with the exception of the tensions. To determine these we must invoke the inextensibility condition which was assumed at the outset. This takes the form of a constraint on the motion of the line. It requires that the separation between adjacent stations must not change with time. Thus, we write

$$(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 = \ell_{j-1/2}^2 = \text{const.} \qquad (2.6)$$

This holds for each segment of the line, and we require that the corresponding set of tensions, $T_{j-1/2}$, take on values such that the resulting solution of

the equations of motion will be consistent with eq. (2.6). Because of the implicit nature of this condition, we are led to a system of algebraic equations for the determination of the proper tensions. At the extremities of the line (j = 0 and j = s) x_j and y_j must be obtained from the boundary conditions, namely:

$$\begin{aligned} x_0 &= x_0(t) \\ y_0 &= y_0(t) \\ x_s &= x_s(t) \\ y_s &= y_s(t) \end{aligned} \qquad (2.7)$$

These are given as functions of time, and permit the introduction of any desired types of driving motions.

Finally, to complete the formulation of the problem a set of initial conditions must be given for each station on the line. Since the equations of motion are of the second order, it is necessary to specify both the coordinates and the velocities at t = 0. That is,

$$\begin{aligned} x_j(0) &= x_j^0 \qquad 0 < j < s \\ y_j(0) &= y_j^0 \qquad " \\ \dot{x}_j(0) &= \dot{x}_j^0 \qquad " \\ \dot{y}_j(0) &= \dot{y}_j^0 \qquad " \end{aligned} \qquad (2.8)$$

where the superscript index "0" is used to designate a value at the origin in time.

3. SOLUTION OF EQUATIONS BY FINITE DIFFERENCES

A. General Description of Computational Procedure

The equations governing the motion of a cable, as derived in the last section, are summarized here. The basic equations of motion, equations (2. 2), are repeated for convenience,

$$I_j \ddot{x}_j - K_j \ddot{y}_j = T_{j+\frac{1}{2}} \cos\theta_{j+\frac{1}{2}} - T_{j-\frac{1}{2}} \cos\theta_{j-\frac{1}{2}} + X_j, \qquad j = 1, 2, \ldots S-1$$

$$-K_j \ddot{x}_j + J_j \ddot{y}_j - T_{j+\frac{1}{2}} \sin\theta_{j+\frac{1}{2}} - T_{j-\frac{1}{2}} \sin\theta_{j-\frac{1}{2}} + Y_j, \qquad j = 1, 2, \ldots S-1 \tag{2. 2}$$

where

a) S is the number of junction points

b) I_j, K_j, J_j are given in equation (2. 1) and are functions of the physical properties of the cable and of position only

c) X_j, Y_j are given by equations (2. 3), (2. 4), and (2. 5) and are functions of the physical properties of the cable and of position and velocity.

In addition the motion is governed by the condition of inextensibility of the cable, equation (2. 6),

$$(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 = l_{j-\frac{1}{2}}^2 = \text{const}, \qquad j = 1, 2, \ldots S. \tag{2. 6}$$

The differentiated (with respect to time) forms of this relation

$$(x_j - x_{j-1})(\dot{x}_j - \dot{x}_{j-1}) + (y_j - y_{j-1})(\dot{y}_j - \dot{y}_{j-1}) = 0, \qquad j = 1, 2 \ldots S \tag{3. 1}$$

$$(x_j - x_{j-1})(\ddot{x}_j - \ddot{x}_{j-1}) + (y_j - y_{j-1})(\ddot{y}_j - \ddot{y}_{j-1}) + (\dot{x}_j - \dot{x}_{j-1})^2 + (\dot{y}_j - \dot{y}_{j-1})^2 = 0 \quad j = 1, 2 \ldots S \tag{3. 2}$$

are also used in the computation.

For numerical solution by finite difference methods the following finite difference equivalents are used,

$$\dot{x}_j^{n+\frac{1}{2}} = \frac{x_j^{n+1} - x_j^n}{\Delta t}, \qquad \dot{y}_j^{n+\frac{1}{2}} = \frac{y_j^{n+1} - y_j^n}{\Delta t}, \qquad j = 1, 2 \ldots S-1; \tag{3. 3}$$

$$\ddot{x}_j^n = \frac{x_j^{n+1} - 2x_j^n + x_j^{n-1}}{(\Delta t)^2}, \qquad \ddot{y}_j^n = \frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{(\Delta t)^2}, \quad j = 1, 2 \ldots S-1. \tag{3. 4}$$

It is assumed that the boundary and initial conditions are known. These are given in equations (2. 7) and (2. 8), respectively. The system of equations summarized above, consisting of equations (2. 2), (2. 6), (2. 7), (2. 8), (3. 1), (3. 2), (3. 3), (3. 4) with the auxiliary equations (2. 1), (2. 3), (2. 4), (2. 5) completely describe the motion of the cable.

The computational procedure, as developed in detail in the remainder of this section, consists of an algorithm to determine the values x_j^{n+1}, y_j^{n+1}, $\dot{x}_j^{n+\frac{1}{2}}$, $\dot{y}_j^{n+\frac{1}{2}}$ (at time $t = t^{n+1} = t^n + \Delta t$ and $t^{n+\frac{1}{2}} = t^n + \frac{\Delta t}{2}$) from known values x_j^n, y_j^n, $\dot{x}_j^{n-\frac{1}{2}}$ $\dot{y}_j^{n-\frac{1}{2}}$ (at time $t = t^n$ and $t^{n-\frac{1}{2}}$).

It is convenient to divide this algorithm in two phases, or steps. In the first step tentative (or starting) values for x_j^{n+1}, y_j^{n+1}, $\dot{x}_j^{n+\frac{1}{2}}$, $\dot{y}_j^{n+\frac{1}{2}}$ are obtained. In the second step improved solutions are obtained.

Step 1. Using equations (2. 2) and (3. 2) (3S-2 equations) we compute the (3S-2) unknown variables $\ddot{x}_j^n$, $\ddot{y}_j^n$ (j = 1, 2, ... S-1) and $T_{j-\frac{1}{2}}^n$ (j = 1, 2, ... S). We now use equations (3. 3) and (3. 4) (4S-4 equations) to compute the (4S-4) variables at the next time step x_j^{n+1}, y_j^{n+1}, $\dot{x}_j^{n+\frac{1}{2}}$, $\dot{y}_j^{n+\frac{1}{2}}$ (j = 1, 2...). These are considered only tentative values (denoted in subsequent text by use of the tilde).

Step 2. To obtain the improved values of the tensions $T_{j-\frac{1}{2}}^{n}$ (j = 1, 2, ... S) and the quantities $\ddot{x}_j^n$, $\ddot{y}_j^n$, x_j^{n+1}, y_j^{n+1}, $\dot{x}_j^{n+\frac{1}{2}}$, $\dot{y}_j^{n+\frac{1}{2}}$ (a total of (7S-6) quantities) we use the system of equations (2. 2), (2. 6) and (3. 3), (3. 4), consisting of (7S-6) equations. However, since equations (2. 6) are not linear but quadratic in the unknowns x_j, y_j an explicit solution is impractical to obtain. For this reason a computation algorithm based on the Newton-Raphson method of successive approximations is developed. A detailed discussion of the computation procedure used in this problem is given in the sections which follow.

B. Determination of Tentative Values of Tensions

In summary, the method of solution at each time step involves in the first phase, 1) the determination of a tentative (but consistent) set of tensions $T_{j-1/2}$ for all segments, and 2) numerical integration of the equations of motion to predict x_j and y_j one step ahead; in the second phase 3) evaluation of the discrepancies in the constraint equations from which a set of first order corrections to the tensions can be obtained, and 4) integration of the equations a second time to obtain corrected values of the coordinates.

The system of equations (2. 2) may be regarded as a set of (2S-2) linear equations in the variables $\ddot{x}_j$, $\ddot{y}_j$ (accelerations) and may be solved directly for these variables. If we designate

$$L_j = (\Delta t)^2 I_j / (I_j J_j - K_j^2)$$

$$M_j = (\Delta t)^2 J_j / (I_j J_j - K_j^2)$$

$$N_j = (\Delta t)^2 K_j / (I_j J_j - K_j^2);$$

then the equations of motion (2. 2) can be reduced to:

$$\ddot{x}_j = \left[R_j T_{j+1/2} - P_j T_{j-1/2} + U_j\right] / (\Delta t)^2$$

$$\ddot{y}_j = \left[S_j T_{j+1/2} - Q_j T_{j-1/2} + V_j\right] / (\Delta t)^2$$

where: $P_j = M_j \cos\theta_{j-1/2} + N_j \sin\theta_{j-1/2}$

$Q_j = N_j \cos\theta_{j-1/2} + L_j \sin\theta_{j-1/2}$

$R_j = M_j \cos\theta_{j+1/2} + N_j \sin\theta_{j+1/2}$

$S_j = N_j \cos\theta_{j+1/2} + L_j \sin\theta_{j+1/2}$

$U_j = M_j X_j + N_j Y_j$

$V_j = N_j X_j + L_j Y_j$

We observe that equation (3. 2) involves positions, velocities, and accelerations. As is often the case with finite difference procedures, it proves to be convenient to compute positions and accelerations at the mesh points while velocities are found at the mid-points in time. For this reason we shall use a modified form obtained by evaluating this equation at $t = t^n$ and $t = t^{n-1}$, and then adding the two results together, namely,

$$(x_j^n - x_{j-1}^n)(\ddot{x}_j^n - \ddot{x}_{j-1}^n) + (y_j^n - y_{j-1}^n)(\ddot{y}_j^n - \ddot{y}_{j-1}^n) + (x_j^{n-1} - x_{j-1}^{n-1})(\ddot{x}_j^{n-1} - \ddot{x}_{j-1}^{n-1})$$

$$+ (y_j^{n-1} - y_{j-1}^{n-1})(\ddot{y}_j^{n-1} - \ddot{y}_{j-1}^{n-1}) + 2(\Delta t)^{-2}\left[(x_j^n - x_j^{n-1}) - (x_{j-1}^n - x_{j-1}^{n-1})\right]^2$$

$$+ 2(\Delta t)^{-2}\left[(y_j^n - y_j^{n-1}) - (y_{j-1}^n - y_{j-1}^{n-1})\right]^2 = 0; \qquad (3.6)$$

in which we have used the approximations,

$$(\dot{x}_j^n - \dot{x}_{j-1}^n)^2 + (\dot{x}_j^{n-1} - \dot{x}_{j-1}^{n-1})^2 \approx 2(\dot{x}_j^{n-\frac{1}{2}} - \dot{x}_{j-1}^{n-\frac{1}{2}})^2$$

$$\approx 2\left[(x_j^n - x_j^{n-1})/\Delta t - (x_{j-1}^n - x_{j-1}^{n-1})/\Delta t\right]^2$$

$$(\dot{y}_j^n - \dot{y}_{j-1}^n)^2 + (\dot{y}_j^{n-1} - \dot{y}_{j-1}^{n-1})^2 \approx 2(\dot{y}_j^{n-\frac{1}{2}} - \dot{y}_{j-1}^{n-\frac{1}{2}})^2$$

$$\approx 2\left[(y_j^n - y_j^{n-1})/\Delta t - (y_{j-1}^n - y_{j-1}^{n-1})/\Delta t\right]^2.$$

Note that equations (3.6) are linear in the accelerations. Likewise equations (3.5) are linear in the tensions. Consequently, when these expressions are substituted for the acceleration components in the constraint equations (3.6), we obtain a set of conditions which are linear in the tensions, namely,

$$E^n_{j-1/2}\,\tilde{T}^n_{j-3/2} - F^n_{j-1/2}\,\tilde{T}^n_{j-1/2} + G^n_{j-1/2}\,\tilde{T}^n_{j+1/2} + E^{n-1}_{j-1/2}\,T^{n-1}_{j-3/2}$$

$$- F^{n-1}_{j-1/2}\,T^{n-1}_{j-1/2} + G^{n-1}_{j-1/2}\,T^{n-1}_{j+1/2} + H^n_{j-1/2} + H^{n-1}_{j-1/2} + 2\left[(x^n_j - x^{n-1}_j) - (x^n_{j-1} - x^{n-1}_{j-1})\right]^2$$

$$+ 2\left[(y^n_j - y^{n-1}_j) - (y^n_{j-1} - y^{n-1}_{j-1})\right]^2 = 0 \qquad (3.7)$$

where: $E^n_{j-1/2} = (x^n_j - x^n_{j-1})\,P^n_{j-1} + (y^n_j - y^n_{j-1})\,Q^n_{j-1}$

$$F^n_{j-1/2} = (x^n_j - x^n_{j-1})\,(P^n_j + R^n_{j-1}) + (y^n_j - y^n_{j-1})\,(Q^n_j + S^n_{j-1})$$

$$G^n_{j-1/2} = (x^n_j - x^n_{j-1})\,R^n_j + (y^n_j - y^n_{j-1})\,S^n_j$$

$$H^n_{j-1/2} = (x^n_j - x^n_{j-1})\,(U^n_j - U^n_{j-1}) + (y^n_j - y^n_{j-1})\,(V^n_j - V^n_{j-1})$$

Now assume that the solution is correct up to $t = t^n$. Then all quantities in (3.7) can be evaluated at once except for $\tilde{T}^n_{j-3/2}$, $\tilde{T}^n_{j-1/2}$ and $\tilde{T}^n_{j+1/2}$. The tentative values of the tensions — signified by the tildes — are determined by the following system of equations:

$$\begin{Bmatrix} -F^n_{0.5} & G^n_{0.5} & & & \\ E^n_{1.5} & -F^n_{1.5} & G^n_{1.5} & & \\ & E^n_{2.5} & -F^n_{2.5} & G^n_{2.5} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & E^n_{s-1.5} & -F^n_{s-1.5}\ G_{s-1.5} \\ & & & & E^n_{s-0.5} - F^n_{s-0.5} \end{Bmatrix} \begin{Bmatrix} \tilde{T}^n_{0.5} \\ \tilde{T}^n_{1.5} \\ \tilde{T}^n_{2.5} \\ \vdots \\ \tilde{T}^n_{s-1.5} \\ \tilde{T}^n_{s-0.5} \end{Bmatrix} = \begin{Bmatrix} -\Psi^n_{0.5} \\ -\Psi^n_{1.5} \\ -\Psi^n_{2.5} \\ \vdots \\ -\Psi_{s-1.5} \\ -\Psi_{s-0.5} \end{Bmatrix} \qquad (3.8)$$

where $\Psi^n_{j-1/2} = E^{n-1}_{j-1/2}\,T^{n-1}_{j-3/2} - F^{n-1}_{j-1/2}\,T^{n-1}_{j-1/2} + G^{n-1}_{j-1/2}\,T^{n-1}_{j+1/2} + H^{n-1}_{j-1/2}$

$$+ H^n_{j-1/2} + 2\left[(x^n_j - x^{n-1}_j) - (x^n_{j-1} - x^{n-1}_{j-1})\right]^2 + 2\left[(y^n_j - y^{n-1}_j) - (y^n_{j-1} - y^{n-1}_{j-1})\right]^2$$

In general, we can write: (for $1 \le j \le s$)

$$E^n_{j-1/2}\,\tilde{T}^n_{j-3/2} - F^n_{j-1/2}\,\tilde{T}^n_{j-1/2} + G^n_{j-1/2}\,\tilde{T}^n_{j+1/2} + \Psi^n_{j-1/2} = 0 \qquad (3.9)$$

with the conditions: $E^n_{0.5} = G^n_{s-1/2} = 0$ for all n

Also P^n_0, Q^n_0, R^n_0, S^n_0 and P^n_s, Q^n_s, R^n_s, $S^n_s = 0$ for all n.

and $U^n_0 = (\Delta t)^2\,\ddot{x}^n_0$ and $U^n_s = (\Delta t)^2\,\ddot{x}^n_s$ for all n.

$V^n_0 = (\Delta t)^2\,\ddot{y}^n_0$ and $V^n_s = (\Delta t)^2\,\ddot{y}^n_s$ for all n.

The matrix of coefficients of the system of equations for $\tilde{T}^n_{j-1/2}$ is a triple diagonal one, and it can be easily reduced to a single linear equation by elimination. Thus, we solve equation (3.9) for $\tilde{T}^n_{j+1/2}$.

$$\tilde{T}^n_{j+1/2} = \frac{F^n_{j-1/2}}{G^n_{j-1/2}}\,\tilde{T}^n_{j-1/2} - \frac{E^n_{j-1/2}}{G^n_{j-1/2}}\,\tilde{T}^n_{j-3/2} - \frac{\Psi^n_{j-1/2}}{G^n_{j-1/2}} \qquad (3.10)$$

Now we express each tension as a linear function of $\tilde{T}^n_{0.5}$ (the tension in the first link) as follows:

$$\begin{aligned} \tilde{T}^n_{j+1/2} &= \alpha^n_{j+1/2}\,\tilde{T}^n_{0.5} + \beta^n_{j+1/2} \\ \tilde{T}^n_{j-1/2} &= \alpha^n_{j-1/2}\,\tilde{T}^n_{0.5} + \beta^n_{j-1/2} \\ \tilde{T}^n_{j-3/2} &= \alpha^n_{j-3/2}\,\tilde{T}^n_{0.5} + \beta^n_{j-3/2} \end{aligned} \qquad (3.11)$$

and we arrive at the following recursion formulas for $\alpha^n_{j+1/2}$ and $\beta^n_{j+1/2}$, n

$$\alpha^n_{j+1/2} = (F^n_{j-1/2}\ \alpha^n_{j-1/2} - E^n_{j-1/2}\ \alpha^n_{j-3/2})\ /G^n_{j-1/2} \tag{3.12}$$

$$\beta^n_{j+1/2} = (F^n_{j-1/2}\ \beta^n_{j-1/2} - E^n_{j-1/2}\ \beta^n_{j-3/2} - \Psi^n_{j-1/2})/G^n_{j-1/2}$$

with the conditions: $\alpha^n_{0.5} = 1, \quad \alpha^n_{-0.5} = 0$ for all n.

$$\beta^n_{0.5} = 0 \quad \beta^n_{-0.5} = 0 \text{ for all n.}$$

Starting with j = 1, we evaluate $\alpha^n_{j+1/2}$ and $\beta^n_{j+1/2}$ recursively up to j = s - 1. We then find $\tilde{T}^n_{0.5}$ from the last equation of the system, (j = s), using the same substitutions as before, that is,

$$\tilde{T}^n_{s-3/2} = \alpha^n_{j-3/2}\ \tilde{T}^n_{0.5} + \beta^n_{j-3/2}$$

$$\tilde{T}^n_{s-1/2} = \alpha^n_{j-1/2}\ \tilde{T}^n_{0.5} + \beta^n_{j-1/2}$$

The final result is

$$\tilde{T}^n_{0.5} = -\frac{(F^n_{s-1/2}\ \beta^n_{s-1/2} - E^n_{s-1/2}\ \beta^n_{s-3/2} - \Psi^n_{s-1/2})}{(F^n_{s-1/2}\ \alpha^n_{s-1/2} - E^n_{s-1/2}\alpha^n_{s-3/2})} \tag{3.13}$$

C. Method of Determining Improved Tensions

In order to solve eqs. (3.5) numerically, we replace $\ddot{x}^n_j$ and $\ddot{y}^n_j$ by their simplest central difference approximations (equation 3.4), namely,

$$\ddot{x}^n_j = (x^{n+1}_j - 2x^n_j + x^{n-1}_j)\ /\ (\Delta t)^2$$
$$\ddot{y}^n_j = (y^{n+1}_j - 2y^n_j + y^{n-1}_j)\ /\ (\Delta t)^2 \tag{3.14}$$

Now we solve for x^{n+1}_j and y^{n+1}_j, considering these as tentative values subject to a slight modification in order to satisfy a system of constraints. Thus we write

$$\tilde{x}^{n+1}_j = 2x^n_j - x^{n-1}_j - P^n_j \tilde{T}^n_{j-1/2} + R^n_j \tilde{T}^n_{j+1/2} + U^n_j$$
$$\tilde{y}^{n+1}_j = 2y^n_j - y^{n-1}_j - Q^n_j \tilde{T}^n_{j-1/2} + S^n_j \tilde{T}^n_{j+1/2} + V^n_j \tag{3.15}$$

The quantities P^n_j, Q^n_j, R^n_j, S^n_j, U^n_j and V^n_j are the same as were used to set up the coefficient matrix for the tensions, and the values for $\tilde{T}^n_{j-1/2}$ and $\tilde{T}^n_{j+1/2}$ are obtained from eqs. (3.11).

Next, we determine the set of corrections $\delta T^n_{j-1/2}$ to be applied to the tensions $\tilde{T}^n_{j-1/2}$ in order that the values of x^{n+1}_j and y^{n+1}_j should also satisfy the inextensibility condition (2.6). For this purpose we define the function

$$\Omega^{n+1}_{j-1/2} = \frac{1}{2}\left[(x^{n+1}_j - x^{n+1}_{j-1})^2 + (y^{n+1}_j - y^{n+1}_{j-1})^2 - \ell^2_{j-1/2}\right] \tag{3.16}$$

which measures the discrepancy in the distance between the extrapolated positions of pairs of adjacent stations. We observe from eqs. (3.15) — with the tildes suppressed — that x^{n+1}_j and y^{n+1}_j are functions of the tensions, that is,

$$x_j^{n+1} = x_j^{n+1}\left\{T_{j-1/2}^n,\ T_{j+1/2}^n\right\}$$

$$y_j^{n+1} \quad y_j^{n+1}\left\{T_{j-1/2}^n,\ T_{j+1/2}^n\right\}$$

Consequently, $\Omega_{j-1/2}^{n+1}$ may also be expressed as a function of the tensions.

This enables us to write the system of constraints to which the tensions are subject as follows:

$$\Omega_{j-1/2}^{n+1} = \Omega_{j-1/2}^{n+1}\left\{T_{j-3/2}^n,\ T_{j-1/2}^n,\ T_{j+1/2}^n\right\} = 0 \quad (1 \le j \le s) \quad (3.17)$$

since $\Omega_{j-1/2}^{n+1}$ vanishes when the inextensibility condition is obeyed.

Now let
$$T_{j-3/2}^n = \tilde{T}_{j-3/2}^n + \delta T_{j-3/2}^n$$

$$T_{j-1/2}^n = \tilde{T}_{j-1/2}^n + \delta T_{j-1/2}^n \quad (3.18)$$

$$T_{j+1/2}^n = \tilde{T}_{j+1/2}^n + \delta T_{j+1/2}^n$$

and expand $\Omega_{j-1/2}^{n+1}$ in a Taylor series about the point $\left\{\tilde{T}_{j-3/2}^n,\ \tilde{T}_{j-1/2}^n,\ \tilde{T}_{j+1/2}^n\right\}$.

Thus, we obtain

$$\Omega_{j-1/2}^{n+1} = \tilde{\Omega}_{j-1/2}^{n+1} + \frac{\partial \tilde{\Omega}_{j-1/2}^{n+1}}{\partial T_{j-3/2}^n}\,\delta T_{j-3/2}^n + \frac{\partial \tilde{\Omega}_{j-1/2}^{n+1}}{\partial T_{j-1/2}^n}\,\delta T_{j-1/2}^n + \frac{\partial \tilde{\Omega}_{j-1/2}^{n+1}}{\partial T_{j+1/2}^n}\,\delta T_{j+1/2}^n \quad (3.19)$$

$$+ \text{ higher order terms.}$$

where: $\tilde{\Omega}_{j-1/2}^{n+1} \equiv \Omega_{j-1/2}^{n+1}\left\{\tilde{T}_{j-3/2}^n,\ \tilde{T}_{j-1/2}^n,\ \tilde{T}_{j+1/2}^n\right\}$

$$= \frac{1}{2}\left[(\tilde{x}_j^{n+1} - \tilde{x}_{j-1}^{n+1})^2 + (\tilde{y}_j^{n+1} - \tilde{y}_{j-1}^{n+1})^2 - \ell_{j-1/2}^2\right]$$

Provided that the tentative values $\tilde{T}_{j-1/2}^n$ are sufficiently close to the correct values $T_{j-1/2}^n$, we may neglect the higher order terms in the expansion (3.19), and thereby obtain a system of s linear equations for the differential corrections $\delta T_{j-1/2}^n$. These equations have the same form as the previous system (3.8) for determining $\tilde{T}_{j-1/2}^n$, namely,

$$\begin{array}{|cccccc|c|c|}
-\tilde{F}_{0.5}^{n+1} & \tilde{G}_{0.5}^{n+1} & & & & & \delta T_{0.5}^n & -\tilde{\Omega}_{0.5}^{n+1} \\
\tilde{E}_{1.5}^{n+1} & -\tilde{F}_{1.5}^{n+1} & \tilde{G}_{1.5}^{n+1} & & & & \delta T_{1.5}^n & -\tilde{\Omega}_{1.5}^{n+1} \\
 & \tilde{E}_{2.5}^{n+1} & -\tilde{F}_{2.5}^{n+1} & \tilde{G}_{2.5}^{n+1} & & & \delta T_{2.5}^n & -\tilde{\Omega}_{2.5}^{n+1} \\
 & & \cdots\cdots & & & & \vdots & \vdots \\
 & & & \tilde{E}_{s-1.5}^{n+1} & -\tilde{F}_{s-1.5}^{n+1} & \tilde{G}_{s-1.5}^{n+1} & \delta T_{s-1.5}^n & -\tilde{\Omega}_{s-1.5}^{n+1} \\
 & & & & \tilde{E}_{s-0.5}^{n+1} & -\tilde{F}_{s-0.5}^{n+1} & \delta T_{s-0.5}^n & -\tilde{\Omega}_{s-0.5}^{n+1}
\end{array} \quad (3.20)$$

the general expression being: (for $1 \le j \le s$)

$$\tilde{E}_{j-1/2}^{n+1}\,\delta T_{j-3/2}^n - \tilde{F}_{j-1/2}^{n+1}\,\delta T_{j-1/2}^n + \tilde{G}_{j-1/2}^{n+1}\,\delta T_{j+1/2}^n + \tilde{\Omega}_{j-1/2}^{n+1} = 0 \quad (3.21)$$

where:
$$\tilde{E}_{j-1/2}^{n+1} = \frac{\partial \tilde{\Omega}_{j-1/2}^{n+1}}{\partial T_{j-3/2}^n} = (\tilde{x}_j^{n+1} - \tilde{x}_{j-1}^{n+1})\left[\frac{\partial \tilde{x}_j^{n+1}}{\partial T_{j-3/2}^n} - \frac{\partial \tilde{x}_{j-1}^{n+1}}{\partial T_{j-3/2}^n}\right]$$

$$+ (\tilde{y}_j^{n+1} - \tilde{y}_{j-1}^{n+1})\left[\frac{\partial \tilde{y}_j^{n+1}}{\partial T_{j-3/2}^n} - \frac{\partial \tilde{y}_{j-1}^{n+1}}{\partial T_{j-3/2}^n}\right]$$

$$-\bar{F}^{n+1}_{j-1/2} = \frac{\partial \bar{\Omega}^{n+1}_{j-1/2}}{\partial T^n_{j-1/2}} = (\bar{x}^{n+1}_j - \bar{x}^{n+1}_{j-1}) \left[\frac{\partial \bar{x}^{n+1}_j}{\partial T^n_{j-1/2}} - \frac{\partial \bar{x}^{n+1}_{j-1}}{\partial T^n_{j-1/2}} \right]$$

$$+ (\bar{y}^{n+1}_j - \bar{y}^{n+1}_{j-1}) \left[\frac{\partial \bar{y}^{n+1}_j}{\partial T^n_{j-1/2}} - \frac{\partial \bar{y}^{n+1}_{j-1}}{\partial T^n_{j-1/2}} \right]$$

$$\bar{G}^{n+1}_{j-1/2} = \frac{\partial \bar{\Omega}^{n+1}_{j-1/2}}{\partial T^n_{j+1/2}} = (\bar{x}^{n+1}_j - \bar{x}^{n+1}_{j-1}) \left[\frac{\partial \bar{x}^{n+1}_j}{\partial T^n_{j+1/2}} - \frac{\partial \bar{x}^{n+1}_{j-1}}{\partial T^n_{j+1/2}} \right]$$

$$+ (\bar{y}^{n+1}_j - \bar{y}^{n+1}_{j-1}) \left[\frac{\partial \bar{y}^{n+1}_j}{\partial T^n_{j+1/2}} - \frac{\partial \bar{y}^{n+1}_{j-1}}{\partial T^n_{j+1/2}} \right]$$

These simplify to:

$$\bar{E}^{n+1}_{j-1/2} = (\bar{x}^{n+1}_j - \bar{x}^{n+1}_{j-1}) P^n_{j-1} + (\bar{y}^{n+1}_j - \bar{y}^{n+1}_{j-1}) Q^n_{j-1}$$

$$\bar{F}^{n+1}_{j-1/2} = (\bar{x}^{n+1}_j - \bar{x}^{n+1}_{j-1}) (P^n_j + R^n_{j-1}) + (\bar{y}^{n+1}_j - \bar{y}^{n+1}_{j-1}) (Q^n_j + S^n_{j-1}) \qquad (3.22)$$

$$\bar{G}^{n+1}_{j-1/2} = (\bar{x}^{n+1}_j - \bar{x}^{n+1}_{j-1}) R^n_j + (\bar{y}^{n+1}_j - \bar{y}^{n+1}_{j-1}) S^n_j$$

with the conditions: $\bar{E}^{n+1}_{0.5} = \bar{G}^{n+1}_{s-1/2} = 0$ for all n, and the quantities P^n_j, Q^n_j, R^n_j and S^n_j being the same as before.

The system (3.20) can be solved in a manner completely analogous to the solution of the system (3.8). Thus, we write

$$\delta T^n_{j-1/2} = \kappa^n_{j-1/2} \, \delta T^n_{0.5} + \lambda^n_{j-1/2} \qquad (3.23)$$

and obtain the following recursion formulas:

$$\kappa^n_{j+1/2} = (\bar{F}^{n+1}_{j-1/2} \kappa^n_{j-1/2} - \bar{E}^{n+1}_{j-1/2} \kappa^n_{j-3/2}) / \bar{G}^{n+1}_{j-1/2}$$

$$\lambda^n_{j+1/2} = (\bar{F}^{n+1}_{j-1/2} \lambda^n_{j-1/2} - \bar{E}^{n+1}_{j-1/2} \lambda^n_{j-3/2} - \bar{\Omega}^{n+1}_{j-1/2}) / \bar{G}^{n+1}_{j-1/2} \qquad (3.24)$$

with the conditions: $\kappa^n_{0.5} = 1$, $\kappa^n_{-0.5} = 0$ for all n.

$\lambda^n_{0.5} = 0$, $\lambda^n_{-0.5} = 0$ " " "

Finally, the last equation of the system (when j = s) enables us to solve for $\delta T^n_{0.5}$. The result is

$$\delta T^n_{0.5} = - \frac{(\bar{F}^{n+1}_{s-1/2} \lambda^n_{s-1/2} - \bar{E}^{n+1}_{s-1/2} \lambda^n_{s-3/2} - \bar{\Omega}^{n+1}_{s-1/2})}{(\bar{F}^{n+1}_{s-1/2} \kappa^n_{s-1/2} - \bar{E}^{n+1}_{s-1/2} \kappa^n_{s-3/2})} \qquad (3.25)$$

D. Computation of New Coordinates.

We can now obtain the corrected values of the tensions in every link. Thus,

$$T^n_{j-1/2} = \tilde{T}^n_{j-1/2} + \delta T^n_{j-1/2}$$

$$= \tilde{T}^n_{j-1/2} + \kappa^n_{j-1/2}\ \delta T^n_{0.5} + \lambda^n_{j-1/2} \qquad (3.26)$$

The corrected values of the coordinates are found using eqs. (3.15) — but this time with the tildes suppressed — namely:

$$x^{n+1}_j = 2x^n_j - x^{n-1}_j - P^n_j T^n_{j-1/2} + R^n_j T^n_{j+1/2} + U^n_j$$

$$y^{n+1}_j = 2y^n_j - y^{n-1}_j - Q^n_j T^n_{j-1/2} + S^n_j T^n_{j+1/2} + V^n_j \qquad (3.27)$$

For solution on an automatic computer it is more convenient to express eqs. (3.27) in terms of corrections to be added to the tentative values of the coordinates. That is,

$$\delta x^{n+1}_j = - P^n_j\ \delta T^n_{j-1/2} + R^n_j\ \delta T^n_{j+1/2}$$

$$\delta y^{n+1}_j = - Q^n_j\ \delta T^n_{j+1/2} + S^n_j\ \delta T^n_{j+1/2} \qquad (3.28)$$

Then the corrected coordinates are given by:

$$x^{n+1}_j = \tilde{x}^{n+1}_j + \delta x^{n+1}_j$$

$$y^{n+1}_j = \tilde{y}^{n+1}_j + \delta y^{n+1}_j \qquad (3.29)$$

These values are now accepted as final. Moreover, as soon as the values of $\Psi^{n+1}_{j-1/2}$ (to be used with eqs. (3.8) for the next time step) have been computed and stored, the cycle of computations is finished and there is no further need to retain the values of P^n_j, Q^n_j, R^n_j, S^n_j and $T^n_{j-1/2}$.

E. Special Form of Equations for Computing First Time Step.

We assume that the velocity components are zero at each station, and obtain the initial coordinates from the equations for static equilibrium of the line. Since $\dot{x}_j^o$ and $\dot{y}_j^o = 0$, eq. (3.2) reduces to

$$(x_j^o - x_{j-1}^o)(\ddot{x}_j^o - \ddot{x}_{j-1}^o) + (y_j^o - y_{j-1}^o)(\ddot{y}_j^o - \ddot{y}_{j-1}^o) = 0 \qquad (3.30)$$

and, on substituting the expressions (3.5), we find that the tensions are subject to the constraint

$$E_{j-1/2}^o \tilde{T}_{j-3/2}^o - F_{j-1/2}^o \tilde{T}_{j-1/2}^o + G_{j-1/2}^o \tilde{T}_{j+1/2}^o + H_{j-1/2}^o = 0 \qquad (3.31)$$

Comparing this with eq. (3.9), we see that

$$\mathfrak{H}_{j-1/2}^o = H_{j-1/2}^o \qquad (3.32)$$

The system of equations (3.8) is then solved in the usual way to get the proper initial tensions $\tilde{T}_{j-1/2}^o$.

To obtain tentative values for the coordinates at $t = t^1$, we make use of their Taylor series expansions about the point $t = t^o$, namely:

$$x_j^1 = x_j^o + (\Delta t)\dot{x}_j^o + \frac{1}{2}(\Delta t)^2 \ddot{x}_j^o + \cdots$$
$$y_j^1 = y_j^o + (\Delta t)\dot{y}_j^o + \frac{1}{2}(\Delta t)^2 \ddot{y}_j^o + \cdots \qquad (3.34)$$

Taking x_j^o and $y_j^o = 0$, and substituting eqs. (3.5) for $\ddot{x}_j^o$ and $\ddot{y}_j^o$, we find

$$\tilde{x}_j^1 = x_j^o + \frac{1}{2}\left[- P_j^o \tilde{T}_{j-1/2}^o + R_j^o \tilde{T}_{j+1/2}^o + U_j^o \right]$$
$$\tilde{y}_j^1 = y_j^o + \frac{1}{2}\left[- Q_j^o \tilde{T}_{j-1/2}^o + S_j^o \tilde{T}_{j+1/2}^o + V_j^o \right] \qquad (3.34)$$

The corrections to the tensions are then determined by the system of equations (3.20) in the usual manner. Finally, the corrections to the coordinates are computed as follows:

$$\delta x_j^1 = \frac{1}{2}\left[- P_j^o \delta T_{j-1/2}^o + R_j^o \delta T_{j+1/2}^o \right]$$
$$\delta y_j^1 = \frac{1}{2}\left[- Q_j^o \delta T_{j-1/2}^o + S_j^o \delta T_{j+1/2}^o \right] \qquad (3.35)$$

and the corrected coordinates are given by:

$$x_j^1 = \tilde{x}_j^1 + \delta x_j^1$$
$$y_j^1 = \tilde{y}_j^1 + \delta y_j^1 \qquad (3.36)$$

4. ANALYSIS OF NUMERICAL STABILITY

In order to obtain a valid solution of the system of partial differential equations (2.2), (2.6), governing the generalized motion of a cable it is necessary to insure the stability (in the sense discussed in References 1, 2, 3)* of the equivalent finite difference system (2.2), (2.6), (3.3), (3.4). In this section we will derive the criteria for stability of this system of equations. We will also show that whereas the system of finite difference equations (2.2), (2.6), (3.3), (3.4) is stable for sufficiently small time intervals Δt, the system (2.2), (3.2), (3.3), (3.4) is always unstable. This characteristic of the latter system has led to the abandonment of this simpler set of equations in favor of the more difficult nonlinear system (2.2), (2.6), (3.3), (3.4).

In order to determine the stability of a system of finite difference equations we study the growth of a small disturbance or perturbation. The conditions for stability are said to be satisfied if the amplitude of a small disturbance, introduced at any time, t, in any of the dependent variables, does not increase exponentially with successive time steps. This condition may be stated as follows:

If $\delta F(s,t)$ and $\delta F(s, t+\Delta t)$ are values of a variation (or perturbation) in any of the dependent variables x, y, T in the system, then it is said to be stable provided $\left|\delta F(s, t+\Delta t)/\delta F(s,t)\right| \leq 1$. We introduce perturbations δx, δy, δT in the independent variables x, y, T, respectively. For the sake of the stability investigation we further assume that e_j is negligible compared to m_j. Substituting in equations (2.2), (2.6), (3.3), and (3.4) we obtain the variational system of equations

* References are listed on page 41.

$$\bar{m}_j \delta\ddot{x}_j = T_{j+1/2}\,\delta\cos\theta_{j+1/2} - T_{j-1/2}\,\delta\cos\theta_{j-1/2} + \cos\theta_{j+1/2}\,\delta T_{j+1/2} - \cos\theta_{j-1/2}\,\delta T_{j-1/2} - \frac{1}{2}\left[D_{j+1/2}\delta\sin\theta_{j+1/2} + D_{j-1/2}\delta\sin\theta_{j-1/2} + \sin\theta_{j+1/2}\,\delta D_{j+1/2} + \sin\theta_{j-1/2}\delta D_{j-1/2}\right]$$

$$\bar{m}_j \delta\ddot{y}_j = T_{j+1/2}\delta\sin\theta_{j+1/2} - T_{j-1/2}\delta\sin\theta_{j-1/2} + \sin\theta_{j+1/2}\delta T_{j+1/2} - \sin\theta_{j-1/2}\,\delta T_{j-1/2} + \frac{1}{2}\left[D_{j+1/2}\delta\cos\theta_{j+1/2} + D_{j-1/2}\delta\cos\theta_{j-1/2} + \cos\theta_{j+1/2}\delta D_{j+1/2} + \cos\theta_{j-1/2}\delta D_{j-1/2}\right] \qquad (4.1)$$

$$\cos\theta_{j+1/2}\delta\cos\theta_{j+1/2} + \sin\theta_{j+1/2}\delta\sin\theta_{j+1/2} = 0$$

where,

$$\delta D_{j+1/2} = -2 f^{D}_{j+1/2}\left|q_{j+1/2}\right|\delta q_{j+1/2}$$

$$\bar{m}_j = m_j + m^{*}_j$$

$$\delta q_{j+1/2} = -\frac{1}{2}\left[(\dot{x}_{j+1} - c) + (\dot{x}_j - c)\right]\delta\sin\theta_{j+1/2} + \frac{1}{2}(\dot{y}_{j+1} + \dot{y}_j)\delta\cos\theta_{j+1/2} - \frac{1}{2}\sin\theta_{j+1/2}(\delta\dot{x}_{j+1} + \delta\dot{x}_j) + \frac{1}{2}\cos\theta_{j+1/2}(\delta\dot{y}_{j+1} + \delta\dot{y}_j);$$

and where,

$$\delta\cos\theta_{j+1/2} = (\delta x_{j+1} - \delta x_j)/\ell_{j+1/2},\ \delta\sin\theta_{j+1/2} = (\delta y_{j+1} - \delta y_j)/\ell_{j+1/2};$$

$$\delta\dot{x}^{n-1/2}_j = (\delta x^n_j - \delta x^{n-1}_j)/\Delta t,\ \delta\dot{y}^{n-1/2}_j = (\delta y^n_j - \delta y^{n-1}_j)/\Delta t;$$

$$\delta\ddot{x}^n_j = (\delta x^{n+1}_j - 2\delta x^n_j + \delta x^{n-1}_j)/(\Delta t)^2,\ \delta\ddot{y}^n_j = (\delta y^{n+1}_j - 2\delta y^n_j + \delta y^{n-1}_j)/(\Delta t)^2$$

We will assume in this analysis that within a small region in the (s, t) plane the coefficients (T_j^n, $\cos\theta_j^n$, D_j^n, etc.) of the variational functions vary only slightly and hence may be treated as constants. We will denote these simply by T, cos θ, D, etc., omitting the subscripts. A solution of the system of equations (4.1) can then be obtained in the form,

$$\delta x_j^n = a\, e^{i\beta j + \alpha n \Delta t}$$
$$\delta y_j^n = b\, e^{i\beta j + \alpha n \Delta t}$$
$$\delta T_j^n = c\, e^{i\beta j + \alpha n \Delta t}$$

where, a, b, c are real constants and α complex. Substituting in equation (4.1) we obtain a system of linear homogeneous equations for the quantities a, b and c which has a non-trival solution provided the determinant of the coefficients is identically zero. After some algebraic simplifications the determinant of the coefficients may be written in the form

$$\begin{vmatrix} F - A\sin\theta & D' + B\sin\theta & \cos\theta \\ -D' + A\cos\theta & F - B\cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = 0 \qquad (4.2)$$

where

$$A = f\,q\left[2i\dot{y}\sin\beta - (t\sin\theta/\Delta t)(1+\cos\beta)(1-\lambda^{-1})\right]$$
$$B = f\,q\left[2i(\dot{x}-c)\sin\beta - (t\cos\theta/\Delta t)(1+\cos\beta)(1-\lambda^{-1})\right]$$
$$D' = iD\sin\beta$$
$$F = \bar{m}\,t\,\xi/(\Delta t)^2 + 4T\sin^2\frac{\beta}{2}$$

and where

$$\lambda = e^{\alpha\Delta t}, \quad \xi = (\lambda - 2 + \lambda^{-1}).$$

Multiplying the elements of the determinant and simplifying we obtain

$$A\sin\theta + B\cos\theta - F = 0 \qquad (4.3)$$

But, $A\sin\theta + B\cos\theta =$

$$f\,|q|\left[(2i\sin\beta)p - (t/\Delta t)(1+\cos\beta)(1-\lambda^{-1})\right]$$

where $p = (x - c)\cos\theta + y\sin\theta$

i.e., the tangential component of the velocity of the cable (relative to the medium).

Substituting in equation (4.3) we finally obtain

$$\bar{m}\,t\,\lambda^2 + \left\{f\,|q|\,\Delta t\left[t(1+\cos\beta) - p\,\Delta t\,(2i\sin\beta)\right] + 4T(\Delta t)^2\sin^2\frac{\beta}{2} - 2\bar{m}\,t\right\}\lambda + \left[\bar{m}\,t - f\,|q|\,t\,\Delta t\,(1+\cos\beta)\right] = 0. \qquad (4.4)$$

Now, comparing the first and second terms of the coefficient of λ we find that the second term is negligibly small provided $2p\,\Delta t \ll t$, i.e., the tangential distance traversed by the cable in one time step is very small compared with the length of the cable segment. Since this is usually the case and, at any rate, can always be satisfied by taking the time step sufficiently small, we will omit this term from our subsequent analysis.

For the case of negligible drag, i.e., f = 0, approximately, we obtain from equation (4.4)

$$\lambda^2 + \left[4T(\sin^2\frac{\beta}{2})(\Delta t)^2/\bar{m}\,t - 2\right]\lambda + 1 = 0. \qquad (4.5)$$

In order for the solution to be stable, the conditions $|\lambda_1| \leq 1$, $|\lambda_2| \leq 1$ must both be satisfied. But if λ_1 is a solution of (4.5) then $\lambda_2 = \frac{1}{\lambda_1}$ is also

a solution. It follows that the conditions for stability can only be satisfied if $|\lambda_2| = \left|\frac{1}{\lambda_1}\right| = |\lambda_1| = 1$. Now, let $\lambda_1 = \cos\gamma + i\sin\gamma$, $\lambda_2 = \cos\gamma - i\sin\gamma = \frac{1}{\lambda_1}$; i.e., $|\lambda_1 + \lambda_2| = |2\cos\gamma| \leq 2$. Again from equation (4.5)

$$\lambda_1 + \lambda_2 = 2 - \frac{4T(\sin^2\frac{\beta}{2})(\Delta t)^2}{\bar{m}\ell} .$$

We thus obtain the inequality

$$\left| 2 - 4T(\sin^2\frac{\beta}{2})(\Delta t)^2/\bar{m}\ell \right| \leq 2$$

or

$$T(\sin^2\frac{\beta}{2})(\Delta t)^2/\bar{m}\ell \leq 1. \qquad (4.6)$$

This requirement is tantamount to the condition,

$$\Delta t \leq \sqrt{\frac{\bar{m}\ell}{T}} = \frac{\ell}{\text{velocity of transverse wave}} .$$

In the more general case, allowing for finite drag, equation (4.4) may be reduced to (after neglecting the second term of the coefficient of λ),

$$\bar{m}\ell\lambda^2 + \left[f|q|\ell\Delta t(1+\cos\beta) + 4T(\Delta t)^2\sin^2\frac{\beta}{2} - 2\bar{m}\ell\right]\lambda + \left[\bar{m}\ell - f|q|\ell\Delta t(1+\cos\beta)\right] = 0 \qquad (4.7)$$

This equation is more difficult to analyze. However, it is possible to show that both $|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$ provided the slightly more stringent conditions

$$\Delta t \leq \sqrt{\frac{\bar{m}\ell}{2T}} \quad \text{, and}$$
$$\Delta t \leq \frac{\bar{m}}{2f|q|} \qquad (4.8)$$

are satisfied.

We will now show that the replacement of equation (2.6) by its differentiated form (3.2) results in an unstable system; and that furthermore, the use of any time interval Δt no matter how small does not change the unstable character of the equations. It will suffice to show that this condition exists in the case when the drag is negligible, i.e., f = 0. The variational equation corresponding to equation (3.2) is,

$$(x_j - x_{j-1})(\delta\dot{x}_j - \delta\dot{x}_{j-1}) + (\dot{x}_j - \dot{x}_{j-1})(\delta x_j - \delta x_{j-1}) + 2(x_j - \dot{x}_{j-1})(\delta\dot{x}_j - \delta\dot{x}_{j-1})$$
$$+ (y_j - y_{j-1})(\delta y_j - \delta y_{j-1}) + (\dot{y}_j - y_{j-1})(\delta y_j - \delta y_{j-1}) + 2(y_j - y_{j-1})(\delta y_j - \delta y_{j-1}) = 0.$$

Substituting appropriate values for δx, δy and neglecting terms containing f the determinant equation (4.2) is replaced by

$$\begin{vmatrix} F & 0 & \cos\theta \\ 0 & F & \sin\theta \\ (x_j - x_{j-1})\xi + (\dot{x}_j - x_{j-1})(\Delta t)^2 + 2(\dot{x}_j - \dot{x}_{j-1})(1-\lambda^{-1})\Delta t & (y_j - y_{j-1})\xi + (\dot{y}_j - \dot{y}_{j-1})(\Delta t)^2 + 2(y_j - \dot{y}_{j-1})(1-\lambda^{-1})\Delta t & 0 \end{vmatrix} = 0 \qquad (4.27)$$

Multiplying the elements of the determinant we obtain

$$F\cos\theta\left[(x_j - x_{j-1})\xi + (\dot{x}_j - \dot{x}_{j-1})(\Delta t)^2 + 2(\dot{x}_j - \dot{x}_{j-1})(1-\lambda^{-1})\Delta t\right]$$
$$+ F\sin\theta\left[(y_j - y_{j-1})\xi + (\dot{y}_j - \dot{y}_{j-1})(\Delta t)^2 + 2(y_j - \dot{y}_{j-1})(1-\lambda^{-1})\Delta t\right] = 0 \qquad (4.9)$$

Equating $\cos\theta = \frac{x_j - x_{j-1}}{\ell_{j-1/2}}$, $\sin\theta = \frac{y_j - y_{j-1}}{\ell_{j-1/2}}$;

and using the relation (first time derivative of equation (2.6)),

$$(x_j - x_{j-1})(\dot{x}_j - \dot{x}_{j-1}) + (y_j - y_{j-1})(\dot{y}_j - \dot{y}_{j-1}) = 0$$

as well as equation (3. 2) we obtain in place of equation (4. 9)

$$F\left\{\lambda^2\xi - \left[(\dot{x}_j - \dot{x}_{j-1})^2 + (\dot{y}_j - \dot{y}_{j-1})^2\right](\Delta t)^2\right\} = 0.$$

Thus, in order to satisfy the stability conditions the following two equations must be satisfied

$$F = 0 \tag{4. 10}$$

and

$$\lambda^2\xi - \left[(\dot{x}_j - \dot{x}_{j-1})^2 + (\dot{y}_j - \dot{y}_{j-1})^2\right](\Delta t)^2 = 0. \tag{4. 11}$$

It can be shown that equation (4. 10) is equivalent to the criterion (4. 6) and is satisfied provided

$$\Delta t \leq \sqrt{\frac{\bar{m}\,\ell}{T}}\ .$$

However, equation (4. 11) can never be satisfied for any finite Δt, since it requires that

$$\xi = \left[(\dot{x}_j - \dot{x}_{j-1})^2 + (\dot{y}_j - \dot{y}_{j-1})^2\right](\Delta t)^2/\lambda^2,$$

a positive quantity. This conclusion follows as a result of the definition $\xi = \lambda - 2 + \lambda^{-1}$. If λ_1 is a root of equation (4. 11), then $\lambda_2 = \frac{1}{\lambda_1}$ is also a root of this equation. As before, it follows that for stability $|\lambda_1| \leq 1$ and $|\lambda_2| = \left|\frac{1}{\lambda_1}\right| \leq 1$. Hence $|\lambda_1| = |\lambda_2| = 1$. Let $\lambda_1 = \cos\gamma + i \sin\gamma$, $\gamma_2 = \cos\gamma - \sin\gamma = \frac{1}{\lambda_1}$; then $\xi = 2(\cos\gamma - 1)$, or $-4 \leq \xi \leq 0$. Thus, to satisfy the stability requirement ξ must lie between 0 and - 4, and consequently is always negative or zero.

In Figure 3 the vertical velocity of the midpoint of a mooring line is plotted as a function of time, both as obtained by the use of the stable (valid)

system of equations (2. 2), (2. 6), (3. 3), (3. 4) and as obtained on the basis of the unstable (invalid) system (2. 2), (3. 2), (3. 3), (3. 4). It will be noticed that at approximately 18 seconds the unstable solution rapidly increases beyond any reasonable limit.

5. RESULTS AND CONCLUSIONS

A number of solutions were carried out for varying wave heights and periods. Several typical solutions are reproduced here for the information of the reader. In Figures 4, 5, and 6, plots are given of the maximum tension attained along the cable as a function of time for wave heights of 6 feet and periods of 12.5 seconds, 7.5 seconds, and 5 seconds, respectively. The periods of the variation in maximum tension correspond to the periods of the forced vibration, as expected. The maximum tension, however, increases in amplitude from 32,250 lb in the case of the 12.5 sec period waves to 38,500 lb for 7.5 sec period waves to 49,500 lb when the period is 5 seconds. In Figure 7 the maximum tension attained for wave heights of 9 ft and a period of 7.5 seconds is plotted. The maximum tension is approximately 60,000 lb as compared with 38,500 lb for the case of 6-ft waves with the same period.

As an experiment to aid in understanding the effect of the drag caused by the presence of the fluid on the motion of the cable, one case was carried out with zero drag (i.e., motion in vacuum). A very interesting motion pattern was obtained which appears not to possess a periodic character. This solution is reproduced in Figure 8.

The successful solution of this problem, as well as a number of others, involving complex nonlinear systems of partial differential equations by the use of high-speed calculators and finite difference methods constitutes, in the opinion of the authors, a major advance in applied mathematics.

Until recently it has been considered unfeasible to obtain numerical solutions for general systems of partial differential equations with the exception of a few isolated simple types of equations whose solutions are known in analytic form. However, the solution of engineering problems, in almost every major field of science, is expressible in terms of systems of partial differential equations. Supersonic and subsonic aerodynamics, nuclear reactor design theory, heat flow, propagation of electromagnetic and acoustic waves are but a few areas which fall in this category. In the past engineers have largely depended on experience and on simplified linearized models of the phenomena under study. In the future, such simplified theoretical models will become less valid - as speeds under consideration increase, stresses become larger, temperatures higher. It may also be expected that experimentation will become more costly, more time consuming, and, at times, unfeasible. It is fortuitous that, at the same time, a new approach appears to be unfolding for the solution of many difficult engineering problems - based on the mathematical representation of the phenomenon and the numerical solution of the resulting unabridged system of equations by the use of high-speed calculators and finite difference methods.

The programming of the various phases of this problem was carried out by Mr. Thomas McFee, of the Applied Mathematics Laboratory, in a most effective manner. The speed and accuracy with which he accomplished this phase of the solution were largely responsible for the success in meeting

the required time schedules. The authors would also like to express their gratitude to Mr. R. T. McGoldrick, of the Structural Mechanics Laboratory, for proposing this problem and for a number of helpful discussions; to Dr. R. Bart, Structural Mechanics Laboratory, for a number of ideas used in setting up the numerical procedure; to Dr. E. H. Kennard, David Taylor Model Basin, and Dr. R. M. Langer, Bureau of Ships, for helpful discussions in connection with the definition of the problem; to Dr. Daniel Shanks, Applied Mathematics Laboratory, for valuable suggestions; and to Miss Corinne Lundgren, Applied Mathematics Laboratory, for assistance in the preparation of the figures.

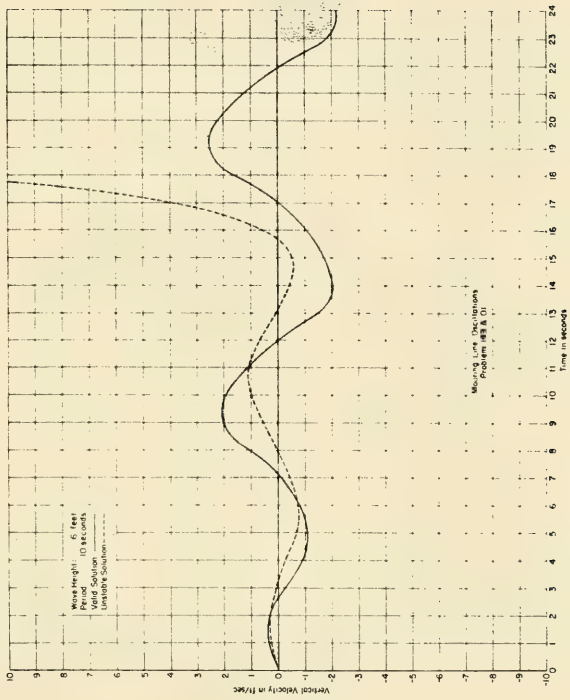

Figure 3

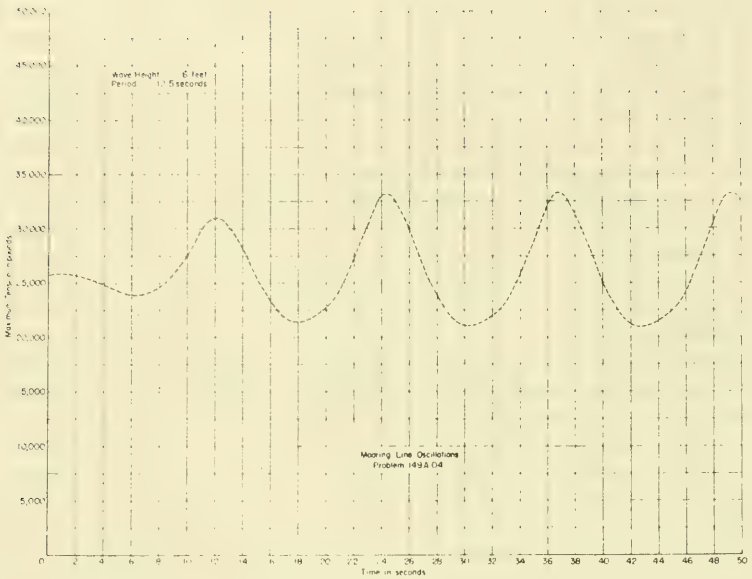

Figure 4

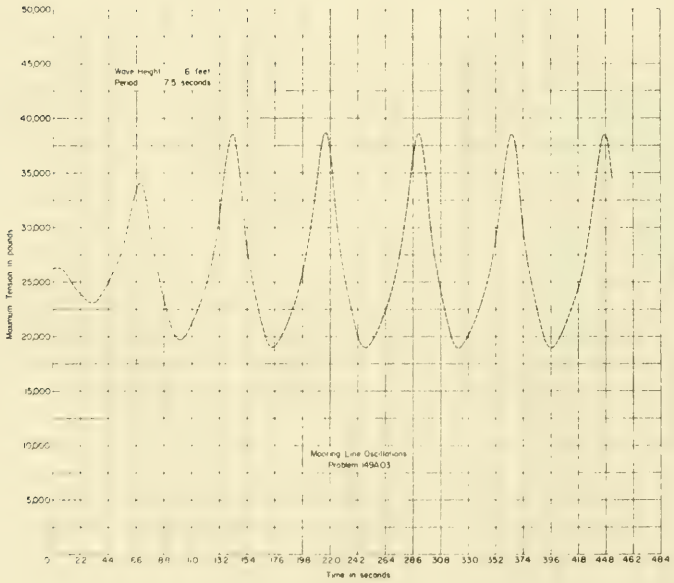

Figure 5

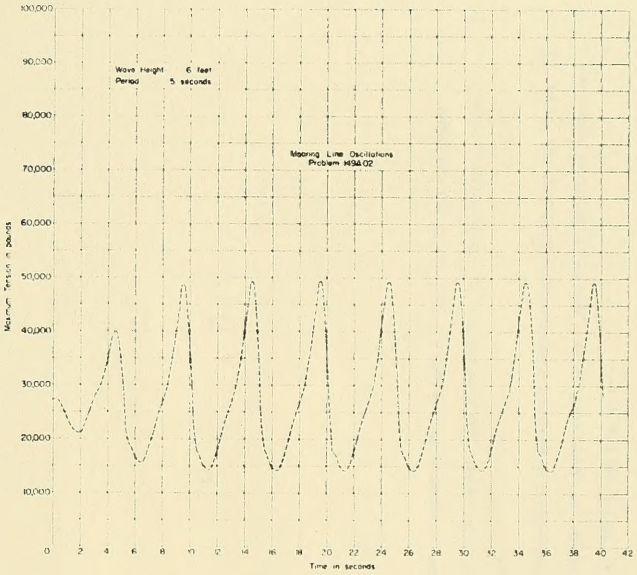

Figure 6

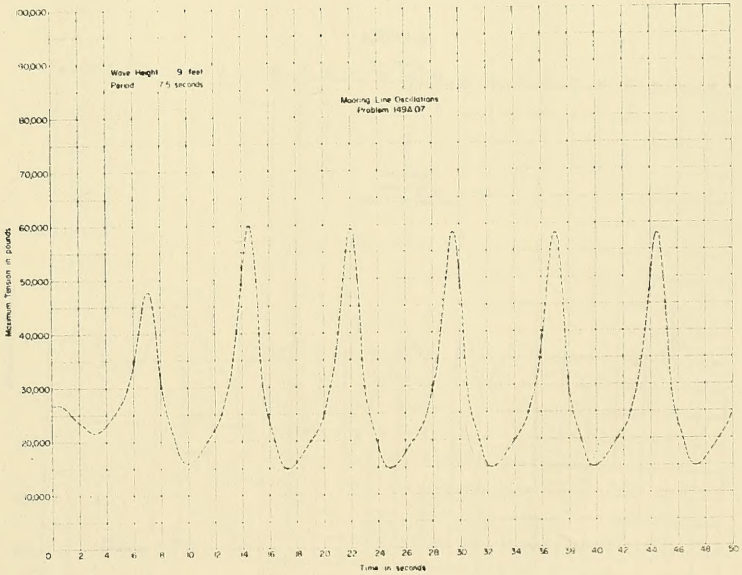

Figure 7

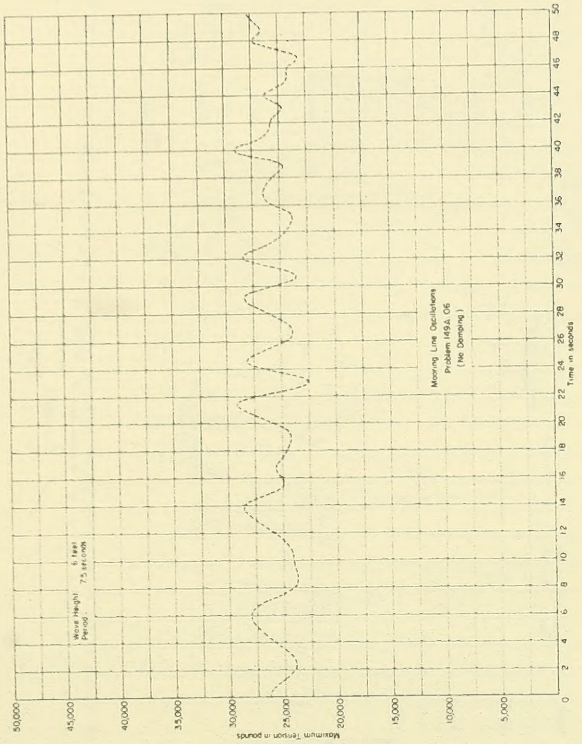

Figure 8

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